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Quantum Field Theories and Critical Phenomena on Defects

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Abstract

We construct and investigate quantum fields induced on a d -dimensional dissipationless defect by bulk fields propagating in a $d + 1$ -dimensional space. All interactions are localized on the defect. We derive a unitary non-canonical quantum field theory on the defect, which is analyzed both in the continuum and on the lattice. The universal critical behavior of the underlying system is determined. It turns out that the $O(N)$ -symmetric φ^4 theory, induced on the defect by massless bulk fields, belongs to the universality class of particular d -dimensional spin models with long-range interactions. On the other hand, in the presence of bulk mass the critical behavior crosses over to the one of d -dimensional spin models with short-range interactions.

Keywords: Quantum Field Theories, Defects, Extra Dimensions, Critical Phenomena

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1 Introduction

Canonical Lagrangian quantum field theory (QFT) dominates our present understanding of elementary particle physics. It is well known however, that in principle one can formulate consistent QFT's without necessarily referring to a Lagrangian and the related canonical formalism. Although may be less attractive in the context of elementary particles, this possibility is relevant in the study of critical phenomena. The enormous progress in conformal field theory (CFT) in the past two decades has shown in fact (see e.g. [1]) that only a small number of such models originate from a canonical Lagrangian.

Inspired by the constant advance in using extra dimensions in QFT, we propose a new class of non-canonical QFT's in d space-time dimensions, induced by canonical fields in $d + 1$ dimensions. The main steps of our construction can be summarized as follows. One starts with a conventional Lagrangian QFT with a local action $A[\Phi_i]$ for the fields $\{\Phi_i\}$ defined on a $d + 1$ -dimensional manifold (bulk space) \mathcal{M} . Then one considers a d -dimensional submanifold $\mathcal{D} \subset \mathcal{M}$, which can be interpreted physically as a defect (impurity) in the bulk. We denote by (x, y) the coordinates of a generic point of \mathcal{M} and assume that the defect \mathcal{D} is recovered for $y \rightarrow 0$. The bulk fields $\{\Phi_i(x, y)\}$ evolve according to the Euler-Lagrange equations following from the bulk action $A_{\mathcal{M}}[\Phi_i]$, supplemented by initial conditions in \mathcal{M} and boundary conditions on \mathcal{D} . The quantum field $\varphi_i(x)$ induced by $\Phi_i(x, y)$ on \mathcal{D} is obtained by performing in appropriate way the limit $y \rightarrow 0$ in $\Phi_i(x, y)$. In spite of the fact that $\{\varphi_i(x)\}$ are non-canonical, their correlation functions define a meaningful local QFT on the defect \mathcal{D} . This theory, which is unitary provided that the defect does not dissipate, will be the goal of our investigation in the present paper. The strategy summarized above consists of two steps. One works first in the bulk, applying standard techniques to the canonical action $A_{\mathcal{M}}[\Phi_i]$. Afterwards, one “projects” the theory on the defect, deriving an effective, non-canonical action $A_{\mathcal{D}}[\varphi_i]$ on \mathcal{D} . An essential aspect of this framework are the interactions, which are assumed to be localized on the defect \mathcal{D} . This idea has been already explored [2] in the context of two-dimensional conformal field theory, where a specific exponential interaction localized at a point has been shown to explain the edge states tunneling in the fractional quantum Hall effect.

The paper is organized as follows. In section 2 we describe how free quantum fields are induced on defects. We discuss the basic properties of the induced fields and derive the effective action on the defect. In section 3 we

investigate the effects of interactions localized on the defect. In particular, we consider the $d+1$ -dimensional theory for an N -component scalar field Φ with $O(N)$ -invariant Φ^4 interaction localized on a d -dimensional defect. We show that the theory can also be formulated in terms of a d -dimensional $O(N)$ vector model defined on the defect and interacting with a $d+1$ -dimensional bulk free field. Using general renormalization-group arguments, we discuss the main features of the universal critical behavior described by the unitary quantum field theory induced on the defect. In section 4 we solve the theory in the large- N limit, both in the continuum and on the lattice. We determine the critical behavior on the defect, and check the scenario put forward in section 3. Finally, section 5 contains our conclusions.

2 Free scalar field induced on a δ -defect

In order to fix the ideas, we consider the simplest example of a bulk scalar field Φ propagating in a $d+1$ -dimensional Minkowski space \mathcal{M} , whose diagonal flat metric has signature $(+, -, \dots, -)$. We adopt the coordinates $(x^0, \dots, x^{d-1}, y) \in \mathcal{M}$ and study a δ -type impurity \mathcal{D} localized on the d -dimensional Minkowski space determined by $y = 0$. The dynamics is defined by the bulk action

$$A_{\mathcal{M}}[\Phi] = -\frac{1}{2} \int_{-\infty}^{\infty} d^d x dy \Phi(x, y) [\square_x - \partial_y^2 + M^2 + 2\mu \delta(y)] \Phi(x, y), \quad (1)$$

where μ characterizes the interaction of Φ with the defect. The variation of (1) gives the equation of motion

$$(\square_x - \partial_y^2 + M^2) \Phi(x, y) = 0, \quad y \neq 0, \quad (2)$$

and the defect boundary conditions

$$\Phi(x, +0) = \Phi(x, -0) \equiv \Phi(x, 0), \quad (3)$$

$$\partial_y \Phi(x, +0) - \partial_y \Phi(x, -0) = 2\mu \Phi(x, 0). \quad (4)$$

The quantum field Φ , satisfying eqs. (2-4) and the conventional equal-time commutation relations, is unique and is fully determined by its two-point function. Performing the quantization, one must take into account that signals propagating in the bulk are both reflected and transmitted [3, 4] by the

impurity. The relative reflection and transmission coefficients in momentum space read

$$R(p) = \frac{-i\mu}{p + i\mu}, \quad T(p) = \frac{p}{p + i\mu} \quad (5)$$

and satisfy unitarity

$$T(p)T(-p) + R(p)R(-p) = 1, \quad (6)$$

$$T(p)R(-p) + R(p)T(-p) = 0. \quad (7)$$

These conditions imply the absence of dissipation on the δ -defect. The coefficients (5) deform the algebra of canonical commutation relations to a reflection-transmission algebra [5], which is the main tool for quantizing equations (2-4). Referring for the details to [6], we focus on the two-point vacuum expectation value of Φ . In the range $\mu \geq 0$ one gets

$$\langle \Phi(x_1, y_1) \Phi(x_2, y_2) \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} E(p; y_1, y_2; \mu) W_{M^2+p^2}(x_{12}). \quad (8)$$

Here $x_{12} \equiv x_1 - x_2$,

$$E(p; y_1, y_2; \gamma) = \theta(y_1)\theta(-y_2)T(p)e^{ipy_{12}} + \theta(-y_1)\theta(y_2)T(-p)e^{ipy_{12}} + \theta(y_1)\theta(y_2) [e^{ipy_{12}} + R(p)e^{ip\tilde{y}_{12}}] + \theta(-y_1)\theta(-y_2) [e^{ipy_{12}} + R(-p)e^{ip\tilde{y}_{12}}] \quad (9)$$

with $\tilde{y}_{12} = y_1 + y_2$ and

$$W_{m^2}(x) = \int_{-\infty}^{\infty} d_d k e^{-ikx} \theta(k^0) 2\pi \delta(k^2 - m^2), \quad d_n k \equiv \frac{d^n k}{(2\pi)^n}, \quad (10)$$

is the standard two-point vacuum expectation value of a free scalar field of mass m in d space-time dimensions.

The quantum field $\varphi(x)$, induced on the δ -defect, is defined by the weak limit

$$\varphi(x) = \sqrt{2} \lim_{y \rightarrow 0} \Phi(x, y) \quad (11)$$

in the Hilbert space of $\Phi(x, y)$. The factor $\sqrt{2}$ is introduced for further convenience. The limit (11) exists [6] and after a straightforward change of variables in (8) one gets

$$\langle \varphi(x_1) \varphi(x_2) \rangle = \int_{M^2}^{\infty} \frac{d\lambda^2}{\pi} \frac{\sqrt{\lambda^2 - M^2}}{\lambda^2 - M^2 + \mu^2} W_{\lambda^2}(x_{12}). \quad (12)$$

Eq. (12) is a Källén-Lehmann spectral representation with density

$$\varrho(\lambda^2) = \theta(\lambda^2 - M^2) \frac{\sqrt{\lambda^2 - M^2}}{\pi(\lambda^2 - M^2 + \mu^2)}, \quad (13)$$

which gives rise to a well-defined generalized free field [7] on \mathcal{D} . In the context of QFT with extra dimensions the induced field φ is a superposition of Kaluza-Klein (KK) modes with mass λ . The positive function ϱ defines the KK measure. Since this measure is polynomially bounded at infinity, φ is a *local* quantum field on the defect.⁴ Obviously it does not satisfy equal-time canonical commutation relations and is therefore non-canonical. Moreover, φ has a continuum mass spectrum with mass gap M .

From (12) one obtains the propagator

$$\tau(x_{12}) \equiv \langle T\varphi(x_1)\varphi(x_2) \rangle = \int_0^\infty d\lambda^2 \varrho(\lambda^2) \Delta_{\lambda^2}(x_{12}), \quad (14)$$

where

$$\Delta_{m^2}(x) = \frac{1}{i} \int_{-\infty}^\infty d_d k \frac{e^{-ikx}}{m^2 - k^2 - i\varepsilon}, \quad (15)$$

is the familiar propagator in d space-time dimensions. The integral in (14) is easily computed and gives for the Fourier transform $\hat{\tau}$ of the propagator the result

$$\hat{\tau}(k) = \frac{1}{i} \frac{1}{\sqrt{M^2 - k^2 - i\varepsilon} + \mu}. \quad (16)$$

It is worth stressing that the poles, that are present in the single KK modes in the complex k_0 -plane, give rise to a cut after the re-summation.

For Euclidean momenta (k_1, \dots, k_d) one gets from (16) the Schwinger function

$$\hat{s}(k_1, \dots, k_d) = i \hat{\tau}(k_0 = -ik_d, k_1, \dots, k_{d-1}) = \frac{1}{\sqrt{M^2 + k^2} + \mu}, \quad (17)$$

which leads to the following Euclidean effective action

$$A_{\mathcal{D}}[\varphi] = \frac{1}{2} \int_{-\infty}^\infty d^d x \varphi(x) \left(\sqrt{M^2 - \Delta} + \mu \right) \varphi(x), \quad (18)$$

⁴The locality properties of induced quantum fields are investigated in [8].

localized on the defect. Eq. (18) reads in momentum space

$$A_{\mathcal{D}}[\varphi] = \frac{1}{2} \int_{-\infty}^{\infty} d_d k \, \widehat{\varphi}(-k) J(k; M, \mu) \widehat{\varphi}(k), \quad (19)$$

$$J(k; M, \mu) \equiv \sqrt{k^2 + M^2} + \mu,$$

where

$$\int_{-\infty}^{\infty} d_d k e^{-ikx} J(k; 0, 0) \sim \frac{1}{r^{d+1}}, \quad r \equiv |x|. \quad (20)$$

In what follows eq. (20) allows to make contact with some previous work [9]-[11] in the context of spin models in statistical mechanics.

The case $-M \leq \mu < 0$ can be considered along the same lines, keeping in mind that the interaction of Φ with the defect produces [6] a defect bound state. It contributes to the Schwinger function, which instead of (17), now takes the form [12]

$$\widehat{s}(k) = \frac{1}{\sqrt{M^2 + k^2} + \mu} + \theta(-\mu) \frac{2|\mu|}{k^2 + M^2 - \mu^2}. \quad (21)$$

In the range $\mu < -M$ the defect boundary state gives imaginary energy contribution to Φ , which leads to an instability. Therefore the free theory (1) is stable and well-defined only when $\mu > -M$.

Let us mention in conclusion that the δ -impurity is actually an element of a whole three-parameter family [13] of dissipationless defects, defined by the boundary conditions

$$\begin{pmatrix} \varphi(x, +0) \\ \partial_y \varphi(x, +0) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \varphi(x, -0) \\ \partial_y \varphi(x, -0) \end{pmatrix}, \quad (22)$$

where $\{a_{ij} \in \mathbb{R} : a_{11}a_{22} - a_{12}a_{21} = 1\}$. The above considerations have a straightforward extension [6] to all of them. For the Källén-Lehmann spectral density one gets in the general case

$$\varrho(\lambda^2) = \theta(\lambda^2 - M^2) \frac{2[a_{12}^2(\lambda^2 - M^2) + a_{11}^2 + 1]\sqrt{\lambda^2 - M^2}}{\pi[a_{12}^2(\lambda^2 - M^2)^2 + (a_{11}^2 + a_{22}^2 + 2)(\lambda^2 - M^2) + a_{21}^2]}, \quad (23)$$

which behaves for $\lambda = 0$ and $\lambda = \infty$ like the density (13). For this reason it is enough to concentrate in what follows on the δ -impurity. Most of our results hold in fact in the general case, because they reflect the low-momentum behavior of the field φ .

3 Φ^4 interactions on δ -defects

In this section we extend the study to the case in which Φ^4 -type interactions are present on the defect and the dynamics is determined by the euclidean bulk action

$$A_{\mathcal{M}}[\Phi] = \int_{-\infty}^{\infty} d^d x dy \left[\frac{1}{2}(\partial\Phi)^2 + \frac{M^2}{2}\Phi^2 + \delta(y) \mu \Phi^2 + \delta(y) \frac{g_0}{4!}(\Phi^2)^2 \right], \quad (24)$$

where Φ is an N -component scalar field. We should mention that surface (boundary) critical phenomena have been widely investigated in the case the Φ^4 -type potential determines also the critical properties in the bulk, see for example the reviews [14, 15]. Here, the defect does not only represent the impurity of the bulk system, but it is also the place where the Φ^4 interaction is localized. As already mention in the introduction, the idea of localizing the interaction on the defect was already considered [2] in the context of two-dimensional conformal field theory and the quantum Hall effect. In the following we discuss the phase diagram and the critical properties on the d -dimensional defect \mathcal{D} , with $d \geq 2$, and especially how they are affected by the presence of a free field in the bulk.

This theory can be generalized by introducing a field φ defined on the defect, and considering the action

$$\begin{aligned} A[\varphi, \Phi] = & \int_{-\infty}^{\infty} d^d x \left[\frac{\kappa}{2}(\partial\varphi)^2(x) + \mu\varphi^2(x) + \frac{g_0}{4!}\varphi^4(x) + \frac{h}{2}[\varphi(x) - \Phi(x, 0)]^2 \right] \\ & + \int_{-\infty}^{\infty} d^d x dy \left[\frac{1}{2}(\partial\Phi)^2(x, y) + \frac{M^2}{2}\Phi^2(x, y) \right], \end{aligned} \quad (25)$$

where $\kappa \geq 0$ and $h > 0$ are real parameters. One can easily check that in the limit $\kappa \rightarrow 0$ and $h \rightarrow \infty$ the theory (24) is recovered (apart from a trivial rescaling of the field). In this model the field φ , constrained on a d -dimensional defect, interacts with a free field Φ propagating in $d + 1$ dimensions. Moreover, as in the case of the standard $O(N)$ -symmetric φ^4 theories, one may also consider a particular limit of the parameters μ and g_0 so that the resulting defect field φ is constrained to have norm one, i.e.

$$\begin{aligned} A[\varphi, \Phi] = & \int_{-\infty}^{\infty} d^d x \left[\frac{\kappa}{2}(\partial\varphi)^2(x) + \frac{h}{2}[\varphi(x) - \Phi(x, 0)]^2 \right] \\ & + \int_{-\infty}^{\infty} d^d x dy \left[\frac{1}{2}(\partial\Phi)^2(x, y) + \frac{M^2}{2}\Phi^2(x, y) \right], \end{aligned} \quad (26)$$

where $\varphi \cdot \varphi \equiv \sum_{i=1}^N \varphi_i \varphi_i = 1$. Note that in the limit $h \rightarrow \infty$, this model can be seen as a free field Φ constrained to have norm one on the defect. The universal critical behavior is expected to remain unchanged under the above changes of the Lagrangian parameters. This can be verified in the large- N limit, see section 4. In particular, the parameters κ and h are expected to be irrelevant from the point of view of the renormalization-group theory, since they do not change the universal behavior at its critical points. They may be set to the values $\kappa = 0$ and $h = \infty$.

The above models can be regularized on a $d + 1$ -dimensional lattice by a straightforward discretization of their Lagrangians. For example, a straightforward discretization of the action (26) is

$$A_{\mathcal{L}} = -\kappa \sum_{x,\mu} \varphi_x \varphi_{x+\mu} + \frac{h}{2} \sum_x (\varphi_x - \Phi_x)^2 + \frac{1}{2} \sum_{z,\nu} (\Phi_z - \Phi_{z+\nu})^2 + \frac{M^2}{2} \sum_z \Phi_z^2, \quad (27)$$

where we set the lattice spacing $a = 1$, x and z indicate respectively the coordinates on the defect and in the bulk, and $\mu = 1, \dots, d$, $\nu = 1, \dots, d + 1$. The corresponding partition function is defined as

$$Z_{\mathcal{L}} = \sum_{\{\Phi\}, \{\varphi\}} \prod_x \delta(\varphi_x^2 - 1) e^{-\beta A_{\mathcal{L}}}, \quad (28)$$

where $\beta = 1/T$, and the coupling T plays the role of temperature. Nontrivial continuum limits are realized at the critical points, where a continuous transitions occur and a length scale diverges in unit of the lattice spacing.

The main features of the phase diagram can be discussed using general renormalization-group arguments. In the case $h = 0$, i.e. when the fields φ and Φ do not interact, the critical properties of the φ field on the defect are those of the d -dimensional N -component nonlinear sigma model. If $d = 1$ the correlation length diverges only when $T \rightarrow 0$ for any N . If $d = 2$, the system undergoes a finite-temperature Ising transition for $N = 1$, a finite- T Kosterlitz-Thouless transition [16] for $N = 2$, while in the case $N \geq 3$ the system becomes critical only in the limit $T \rightarrow 0$, with a length scale ξ that increases exponentially, i.e. $\xi \sim e^{b/T}$, typical of asymptotically free models. For $d = 3$ there is a finite-temperature transition for any N with universal properties characterized by nontrivial power laws.⁵ We have the same scenario for $d = 4$ but with mean-field critical behaviors apart from

⁵The universal properties of three-dimensional $O(N)$ vector models have been deter-

logarithms. For $d > 4$ the behavior is just mean field. See, e.g., refs. [21, 22] for reviews. These critical behaviors remain unchanged for $h > 0$ when $M > 0$, i.e. when M is strictly positive and kept fixed. In this case the large-distance correlations in the bulk decay exponentially, i.e. $\sim e^{-Mr}$, inducing only short-ranged interactions on the defect. They do not change the universal critical behavior of the low modes when the correlation length ξ on the defect is sufficiently large, i.e. when $\xi \gg 1/M$. On the other hand, when the bulk field Φ becomes massless, the large-distance bulk correlations induce long-range interactions on the defect, cf. eq. (20), which can change the critical properties on the defect. These long-range interactions can be inferred from the analysis of section 2. When $M = 0$, integrating out the bulk field Φ in eqs. (26) and (27), we expect to obtain a defect action of the type

$$A_{\mathcal{D}} = \int_{-\infty}^{\infty} d_a k J(k) \hat{\varphi}(-k) \cdot \hat{\varphi}(k), \quad (29)$$

$$J(k) \simeq j_1(k^2)^{1/2} + j_2 k^2 + \dots$$

The critical properties of statistical systems with long-range interactions, such as

$$J(k) = j_s(k^2)^{s/2} + j_2 k^2 + \dots \quad (30)$$

were studied in [9, 10] within an expansion in powers of $\epsilon \equiv 2s - d$ and in [11] in powers of $1/N$. The results of [9]–[11] show that statistical systems with Hamiltonians of the type (29) undergo a continuous transition at finite temperature for any N . The cases $d > 2$, corresponding to $\epsilon < 0$, are in the classical regime [9], where, for any N , the critical behavior of the magnetic susceptibility χ and correlation length ξ are given by [9]

$$\chi \sim \xi \sim t^{-1}, \quad (31)$$

where $t \equiv (T - T_c)/T_c$ is the reduced temperature; therefore the critical exponents are $\gamma = \nu = \eta = 1$. The case $d = 2$ is on the borderline $\epsilon = 0$, and mined by various theoretical methods and experiments. The most precise theoretical estimates of the standard critical exponents have been obtained by lattice techniques. We mention $\nu = 0.63012(16)$, $\eta = 0.03639(15)$ [17] and $\nu = 0.63020(12)$, $\eta = 0.0368(2)$ [18] for the three-dimensional Ising model corresponding to $N = 1$, $\nu = 0.67155(27)$ and $\eta = 0.0380(4)$ [19] for the XY universality class ($N = 2$), and $\nu = 0.7112(5)$ and $\eta = 0.0375(5)$ [20] for the Heisenberg universality class ($N = 3$). In the large- N limit one finds $\nu = 1$ and $\eta = 0$ [21].

multiplicative logarithms correct the classical behavior, [9]

$$\chi \sim \xi \sim t^{-1}(\ln t^{-1})^{\frac{N+2}{N+8}}. \quad (32)$$

We also mention that for $d < 2$, corresponding to $\epsilon > 0$, the critical exponents are nontrivial, indeed

$$\nu = 1 + \frac{N+2}{N+8}\epsilon + O(\epsilon^2), \quad \eta = 1 + O(\epsilon^2). \quad (33)$$

The $O(\epsilon^2)$ terms are reported in refs. [9, 10].

The above-mentioned critical properties suggest that the critical point for $M = 0$ is actually a tricritical point in the T - M plane. Indeed, beside the two standard relevant scaling fields associated with the temperature T and external field H , there is another relevant parameter given by the bulk mass M . Switching M on, the critical behavior crossovers to the more stable one for $M > 0$. We will return to this point in section 4.

We finally note that the critical properties discussed in this section apply also to the more general dissipationless defects defined by eq. (22), since the low-momentum behavior of the corresponding effective Lagrangian remains substantially unchanged.

4 The large- N limit

In this section we show how the scenario put forward in the preceding section is actually realized in the large- N limit, which can be analytically investigated by solving the corresponding saddle-point equations. We first discuss the large- N limit in an appropriate continuum formulation. Then we consider a more rigorous non-perturbative treatment of the bulk theory based on a lattice regularization of the path integral. In other words, we consider a lattice model whose critical behavior is described by the unitary quantum field theory induced on the defect. We present the solution of its large- N limit for $d = 2$. As we shall see, this large- N investigation fully supports the scenario outlined in section 3.

4.1 Continuum theory

We want to study the effects of the Φ^4 interaction localized on the defect, and in particular the universal properties near the critical point at which

the bulk correlation length diverges, i.e. for $M \rightarrow 0$. For this purpose, the results of section 2 suggest to introduce the following effective model defined on the defect by

$$A_{\mathcal{D}}[\varphi] = \int_{-\infty}^{\infty} d^d x \left[\frac{1}{2} \sum_{i=1}^N \varphi_i \left(\sqrt{M^2 - \Delta} + \mu \right) \varphi_i + \frac{g_0}{4!} \left(\sum_{i=1}^N \varphi_i \varphi_i \right)^2 \right]. \quad (34)$$

Note that the above theory is renormalizable in two dimensions, unlike the standard φ^4 model which is renormalizable in four dimensions.

In the following we first determine the critical behavior of the theory (34) at $M = 0$, computing in particular the standard critical exponents and the equation of state in the limit $N \rightarrow \infty$ with fixed $g_0 N \equiv g$. We introduce the auxiliary field λ and a source H for φ_N , writing the partition function in the form

$$Z(H) = \int [d\lambda][d\varphi] e^{-A_{\mathcal{D}}[\varphi, \lambda; H]}, \quad (35)$$

where

$$A_{\mathcal{D}}[\varphi, \lambda; H] = \int_{-\infty}^{\infty} d^d x \left[\frac{1}{2} \sum_{i=1}^N \varphi_i \left(\sqrt{-\Delta} + \lambda \right) \varphi_i + \frac{3\mu}{g_0} \lambda - \frac{3}{2g_0} \lambda^2 - H \varphi_N \right]. \quad (36)$$

Integrating over $\varphi_1, \dots, \varphi_{N-1}$ and setting $\sigma = \varphi_N$, one gets

$$Z(H) = \int [d\lambda][d\sigma] e^{-A_{\mathcal{D}}[\sigma, \lambda; H]}, \quad (37)$$

where

$$A_{\mathcal{D}}[\sigma, \lambda; H] = \int_{-\infty}^{\infty} d^d x \left[\frac{1}{2} \sigma \left(\sqrt{-\Delta} + \lambda \right) \sigma + \frac{3\mu}{g} \lambda - \frac{3}{2g} \lambda^2 + \frac{N-1}{2} \text{Tr} \log \left(\sqrt{-\Delta} + \lambda \right) - H \sigma \right]. \quad (38)$$

The large- N behavior is governed by the uniform ($\sigma = \text{const.}$, $\lambda = \text{const.}$) saddle-point approximation. Varying (38) and rescaling σ and H according to $\sigma \mapsto \sigma \sqrt{N}$ $H \mapsto H \sqrt{N}$, one gets

$$\lambda \sigma = H, \quad (39)$$

$$\sigma^2 - \frac{6}{g} (\lambda - \mu) + \int^{\Lambda} d_d k \frac{1}{|k| + \lambda} = 0. \quad (40)$$

The integral in (40) is regularized by means of an UV cutoff Λ . Moreover, we take $d > 1$ in order to avoid IR instabilities. As expected, eqs. (39,40) strongly resemble their counterparts [21] in the conventional φ^4 theory. The only difference is the term $|k|$ in the denominator of the integrand, which replaces k^2 from the standard φ^4 model. The consequences of this modification are easily analyzed.

Let us consider first the case $H = 0$. In the broken phase $\sigma \neq 0$, one has $\lambda = 0$. Eq. (40) has a solution for σ only if

$$\mu < \mu_c \equiv -\frac{g}{6} \int^\Lambda d_d k \frac{1}{|k|}. \quad (41)$$

Setting

$$t \equiv \frac{6}{g}(\mu - \mu_c), \quad (42)$$

one gets from (40)

$$\sigma^2 = -t \sim (-t)^{2\beta}, \quad (43)$$

which implies $\beta = \frac{1}{2}$.

In the symmetric phase $\sigma = 0$ and $\lambda \equiv m \neq 0$, where m is the inverse correlation length ξ , which determines the large-distance exponential decay of the two-point correlation function of the φ field, i.e. $G(r) \sim e^{-r/\xi}$. Now eq. (40) takes the form

$$\frac{6}{g} + \int^\Lambda d_d k \frac{1}{|k|(|k| + m)} = \frac{t}{m}. \quad (44)$$

From (44) one deduces

$$m \sim t^\nu, \quad \nu = \begin{cases} 1, & d > 2, \\ \frac{1}{d-1}, & 1 < d < 2. \end{cases} \quad (45)$$

From the critical behavior of the two-point correlation function one finds $\eta = 1$. These results agree with the formulas (33) obtained within an $\epsilon \equiv 2-d$ expansion. The case $d = 2$, which is special because the integral in (40) generates the logarithm $\ln \frac{\Lambda}{m}$, will be considered in the next subsection on the lattice. One has mean-field behavior with logarithmic corrections.

We turn now to the case $H \neq 0$. Combining (39) and (40) one gets

$$\sigma^2 + t = \frac{6}{g}m + m \int^\Lambda d_d k \frac{1}{|k|(|k| + m)}. \quad (46)$$

In the domain $1 < d < 2$ it implies

$$H \sim \sigma^\delta f(t\sigma^{-\frac{1}{\beta}}), \quad f(x) = (1+x)^\gamma, \quad (47)$$

$$\delta = \frac{d+1}{d-1}, \quad \gamma = \frac{1}{d-1}, \quad (48)$$

where $f(x)$ is a universal function apart from trivial normalizations (usually one sets $f(0) = 1$ and $f(-1) = 0$ [21]). Note that the critical exponents satisfy the scaling and hyperscaling relations

$$\beta = \frac{\nu}{2}(d-2+\eta), \quad \gamma = \nu(2-\eta), \quad \delta = \frac{d+2-\eta}{d-2+\eta}, \quad (49)$$

in the range $1 < d < 2$. From eq. (47) one may also derive the effective potential (Helmholtz free energy) $V(z)$, i.e. the generator of one-particle irreducible correlation functions of φ at zero momentum, see, e.g., refs. [22, 24]. We obtain

$$V(z) = \frac{3}{d} \left[\left(1 + \frac{d-1}{6} z^2 \right)^{d/(d-1)} - 1 \right], \quad (50)$$

where $z \sim \sigma t^{-\beta}$, which represents the renormalized expectation value of the field φ . In eq. (50) $V(z)$ has been normalized according to $V(z) = \frac{1}{2}z^2 + \frac{1}{24}z^4 + O(z^6)$. For $d = 2$ the effective potential reduces to the simple form

$$V(z) = \frac{1}{2}z^2 + \frac{1}{24}z^4, \quad (51)$$

as in the case of the standard φ^4 model in four dimensions [24].

It is worth noting that the theory (34) induced on a d -dimensional defect with $1 < d < 2$ presents analogies with the standard φ^4 model for $2 < d < 4$, see, e.g., ref. [21]. In both cases we have critical behaviors characterized by nontrivial power laws. Moreover, replacing d with $d/2$ in eqs. (47), (48) and (50), one recovers the corresponding expressions for the large- N limit of the standard $O(N)$ vector models.

As already discussed in section 3, the critical behavior found for $M = 0$ is unstable against the the parameter M , since in the presence of a bulk mass the $M = 0$ critical behavior is not anymore observed. In this case, the models are expected to show the critical behavior of the well known d -dimensional $O(N)$ vector universality class with short-range interactions, i.e.

when $J(k) \simeq k^2$ in eq. (29). Therefore, the critical point for $M = 0$, $T = T_{tc}$, is a tricritical point in the T - M plane. Renormalization-group arguments applied to tricritical points, see, e.g., the review [23], suggest the following generalized scaling hypothesis (for $H = 0$)

$$F_{\text{sing}} \approx t^{\nu} f(Mt^{-\phi}), \quad \chi \approx t^{-\gamma} f_{\chi}(Mt^{-\phi}), \quad (52)$$

where F_{sing} is the singular part of the free energy, χ is the defect magnetic susceptibility; $t \equiv (T - T_{tc})/T_{tc}$, ν , γ , ϕ are the tricritical exponents. In particular ϕ is the crossover exponent associated with the instability direction parametrized by M . The scaling behavior (52) is expected to hold in the critical crossover limit $t, M \rightarrow 0$ keeping $Mt^{-\phi}$ fixed. This issue can be investigated in the large- N limit, starting from the action (34), as we did in this section, but keeping $M \neq 0$ through the various steps to arrive at the saddle-point equation. Setting $H = 0$, we obtain

$$\frac{6}{g}(M - m + \mu) + \int^{\Lambda} d_d k \frac{1}{m - M + \sqrt{M^2 + k^2}} = 0, \quad (53)$$

where now m is related to the zero-momentum correlation function χ (usually called magnetic susceptibility) by $\chi = 1/m$. Defining t by eq. (42), one finds

$$t = \frac{6}{g}(M + m) + \int_{-\infty}^{\infty} d_d k \frac{m - M + \sqrt{M^2 + k^2} - \sqrt{k^2}}{\sqrt{k^2}(m - M + \sqrt{M^2 + k^2})}. \quad (54)$$

Keeping only the leading terms in the critical crossover limit for $1 < d < 2$, we find

$$t \approx m^{d-1} S(M/m), \quad (55)$$

$$S(r) = \int_0^{\infty} d\rho \rho^{(d-2)} \frac{1 - r + \sqrt{r^2 + \rho^2} - \rho}{1 - r + \sqrt{r^2 + \rho^2}}.$$

This result is in agreement with the expected scaling (52), and implies $\phi = 1/(d - 1)$. We also mention that in the classical regime $d > 2$, one finds $\phi = 1$, while for $d = 2$ the classical crossover relations must be corrected due to multiplicative logarithms, as in eq. (32). It is worth mentioning that the relation between χ and ξ in the crossover regime changes: one has $\chi \sim \xi$ in the limiting case $M/m = 0$, while in the opposite limit $M/m = \infty$, $\chi \sim \xi^2$.

4.2 Theory on the lattice

In this section we study the large- N limit of the lattice model defined by the action (27). We consider lattices of size $L^d \times E$ with periodic boundary conditions along all $d + 1$ directions. All dimensional parameters are expressed below in units of the lattice spacing a , which is set to 1 for convenience. To begin with, we integrate out the bulk field Φ in the partition function (28), obtaining a d -dimensional nonlinear sigma model defined on the defect. Straightforward calculations, based on Gaussian integrals, show that its effective action in momentum space reads

$$A_{\text{eff}} = \frac{1}{2} \sum_k [\kappa \hat{k}^2 - B(k; M, h, E)] \varphi_{-k} \cdot \varphi_k, \quad (56)$$

where $\hat{k}^2 = 2 \sum_\mu (1 - \cos k_\mu)$, $k_\mu = 2\pi n_\mu / L$ with $n_\mu = 1, \dots, L$. The function B , following from the integration, is formally given by

$$B(k; M, h, E) = h^2 Q_{1,1}^{-1}, \quad (57)$$

where Q is the $E \times E$ matrix

$$Q = (M^2 + \hat{k}^2) \mathbb{I} + \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & \ddots & \ddots & \ddots & \\ 0 & \ddots & & & \\ \ddots & & & & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix} + h \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & \\ \vdots & & & \end{pmatrix}. \quad (58)$$

We write the partition function as

$$Z_{\mathcal{L}} = \sum_{\{\varphi\}} \prod_x \delta(\varphi_x^2 - 1) e^{-\beta N A_{\text{eff}}}. \quad (59)$$

In the large- N limit keeping β fixed, the solution is given by the saddle-point equation, which, in the limit $L \rightarrow \infty$, takes the form

$$\beta = \int_{-\pi}^{\pi} d_d k \frac{1}{m + \kappa \hat{k}^2 + [B(0; M, h, E) - B(k; M, h, E)]}. \quad (60)$$

Here m is related to the magnetic susceptibility χ by $m = (\beta \chi)^{-1}$. In the large- N limit, $\chi \sim \xi$ for $M = 0$, while $\chi \sim \xi^2$ for $M > 0$. According

to eq. (57), in order to get the function B , we must evaluate the 1,1 matrix element of the inverse matrix of Q given in Eq. (58). In the limit $E \rightarrow \infty$ one finds

$$B(k; M, h, \infty) = -\frac{h^2}{h + [4M^2 + 4\hat{k}^2 + (M^2 + \hat{k}^2)^2]^{1/2}}. \quad (61)$$

This result is derived in the appendix.

In the following we simplify the calculation by fixing $\kappa = 0$ and $h = \infty$, but we have also checked that the universal critical behavior remains the same when choosing generic values $\kappa, h > 0$. In terms of

$$\begin{aligned} R(k; M) &\equiv B(0; M, \infty, \infty) - B(k; M, \infty, \infty) \\ &= \left[4M^2 + 4\hat{k}^2 + (M^2 + \hat{k}^2)^2\right]^{1/2} - (4M^2 + M^4)^{1/2}, \end{aligned} \quad (62)$$

the large- N saddle-point equation becomes

$$\beta = \int_{-\pi}^{\pi} d_d k \frac{1}{m + R(k; M)}. \quad (63)$$

A continuous transition occurs when the defect length scale diverges, i.e. $m \rightarrow 0$.

Let us now consider the case $d = 2$. One can easily check that if $M > 0$ the integral (63) diverges when $m \rightarrow 0$, as expected since when $M > 0$ the effective action represents a particular regularization of the 2- d nonlinear sigma model, which does not have finite-temperature transitions according to the Mermin-Wagner theorem [25]. In this case, the magnetic susceptibility and the correlation length diverges in the limit $\beta \rightarrow \infty$, as

$$\chi \sim \xi^2 \sim e^{4\pi C(M)\beta}, \quad C(M) = \frac{2 + M^2}{M\sqrt{4 + M^2}}, \quad (64)$$

which is the same behavior of the standard 2- d $O(N)$ spin model with $N > 2$ and nearest-neighbor interactions, apart from a trivial normalization of the inverse temperature β . On the other hand, when $M = 0$ a transition occurs at finite temperature, since the integral (63) is finite for $m = 0$. Indeed, one finds

$$\beta_c = \int_{-\pi}^{\pi} d_2 k \frac{1}{R(k; 0)} = 0.252731 \quad (65)$$

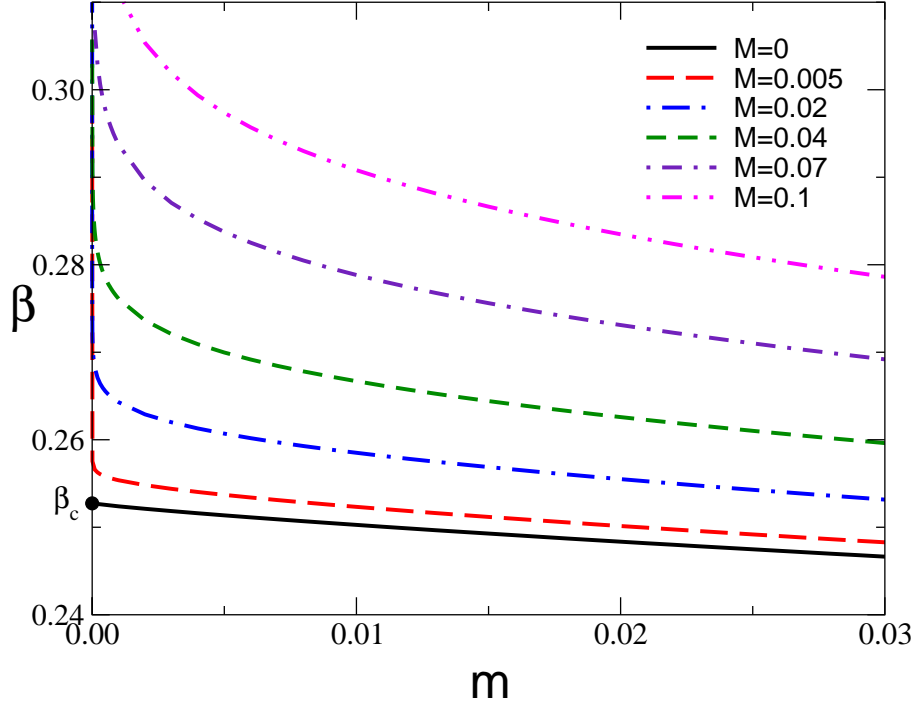


Figure 1: The inverse temperature β versus m for several values of M .

and

$$\chi \sim \xi \sim t^{-1} \ln t^{-1}, \quad (66)$$

where $t \equiv (\beta_c - \beta)/\beta_c$. Note that in this case the Mermin-Wagner theorem [25] does not apply since long-range spin interactions are present.

As already discussed, the critical point T_c for $M = 0$ is actually a tri-critical point. Switching M on, the critical behavior crossovers to the more stable critical behavior for $M > 0$ given by eq. (64). This is shown in Fig. 1, where the function β is plotted against m for various values of M . The curve for $M = 0$ is approached by those for $M > 0$ when $M \rightarrow 0$, except in a small region Δ close to $m = 0$, where they suddenly depart from the $M = 0$ curve and diverge for $m \rightarrow 0$. We note that Δ scales as $\Delta \sim M$ when $M \rightarrow 0$. This is consistent with the expectation that the crossover exponent ϕ is given by $\phi = 1$, apart from logarithmic corrections. We finally mention that the relation among the quantities $\chi = 1/(\beta m)$, t and M can be derived from

equation

$$t \equiv \frac{\beta_c - \beta}{\beta_c} = 1 - \frac{1}{\beta_c} \int_{-\pi}^{\pi} d_2 k \frac{1}{m + R(k; M)}. \quad (67)$$

5 Conclusions

In conclusion, we have shown how non-canonical quantum fields φ_i can be generated in d dimensions using the interaction of canonical fields Φ_i with a δ -type (or more general dissipationless) defect in $d + 1$ dimensions. The fields φ_i , which propagate on the defect, define a unitary quantum field theory there. In order to clarify its basic features, we analyzed the case in which an N -component scalar field Φ_i in $d + 1$ dimensions is subject to a Φ^4 -type interaction localized on a d -dimensional defect. This theory can also be formulated in terms of a d -dimensional $O(N)$ vector model, defined on the defect and interacting with a $d + 1$ -dimensional bulk free field.

General renormalization-group arguments and calculations in the large- N limit allow to get a rather complete picture of the critical behavior of the field theory induced on the defect. The large- N limit of the theory was studied within its continuum formulation and also for statistical systems representing a class of lattice regularizations. The main results can be summarized as follows. When the bulk fields Φ_i are massless, the induced φ^4 theory belongs to the universality class of a specific statistical spin model with long-range interactions, behaving as $\sim 1/r^{d+1}$. The former provides therefore a field theoretical description of the universal critical properties of the latter. The presence of a nonzero bulk mass M causes an interesting crossover to the critical behavior of the standard $O(N)$ spin model with short-range interactions. We argue that the critical point T_{tc} at $M = 0$ is actually a tricritical point in the T - M -plane. Following renormalization-group theory applied to tricritical points, one can then consider a critical crossover limit, defined when $t \sim T - T_{tc} \rightarrow 0$ and $M \rightarrow 0$. This presents a universal scaling behavior, which can be studied within the continuous theory with non-vanishing bulk mass M .

The generalization of this work to models involving gauge interactions opens interesting new possibilities and deserves further investigation.

A Some useful formulas

In this appendix we provide a few details on the derivation of eq. (61). In order to determine the function B from eq. (57), one needs to evaluate the matrix element 1,1 of the inverse of a $n \times n$ matrix of the type

$$A_n = \begin{pmatrix} a & -1 & 0 & \cdots & -1 \\ -1 & b & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & -1 \\ -1 & 0 & \cdots & -1 & b \end{pmatrix}. \quad (68)$$

We are interested in the case $a > b \geq 2$. This can be done by using the formula

$$[A_n^{-1}]_{1,1} = \frac{\det A_n^{1,1}}{\det A_n}, \quad (69)$$

where $A_n^{1,1}$ indicates the minor matrix corresponding to the 1,1 matrix element. Let us introduce the $n \times n$ matrix

$$B_n = \begin{pmatrix} b & -1 & 0 & \cdots \\ -1 & b & \ddots & \ddots \\ 0 & \ddots & \ddots & -1 \\ \vdots & \ddots & -1 & b \end{pmatrix}. \quad (70)$$

Then the following relation holds

$$\begin{aligned} \det A_n^{1,1} &= \det B_{n-1}, \\ \det A_n &= a \det B_{n-1} - 2 \det B_{n-2} - 2. \end{aligned} \quad (71)$$

In order to compute the determinant of the matrix B_n , we note that

$$\det B_1 = b, \quad \det B_2 = b^2 - 1, \quad (72)$$

and the recursive formula

$$\det B_n = b \det B_{n-1} - \det B_{n-2}. \quad (73)$$

In the large- n limit and for $b > 2$, $\det B_n$ diverges and

$$\lim_{n \rightarrow \infty} \frac{\det B_{n-1}}{\det B_n} = \frac{b - \sqrt{b^2 - 4}}{2} < 1. \quad (74)$$

This allows us to derive the formula

$$\begin{aligned}\lim_{n \rightarrow \infty} [A_n^{-1}]_{1,1} &= \lim_{n \rightarrow \infty} \left(a - 2 \frac{\det B_{n-2}}{\det B_{n-1}} - \frac{2}{\det B_{n-1}} \right)^{-1} \\ &= \frac{1}{a - b + \sqrt{b^2 - 4}},\end{aligned}\tag{75}$$

which was used to obtain eq. (61).

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